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# On Generalizes Derivations And commutativity Of Centrally Prime Rings

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**Abstract**

In this paper, we will study some important results relating to the concept of generalized derivation in centrally prime ring to be commutative.

## Introduction

Larsen and McCarthy in [6] introduced the notion of multiplicative system. Let $R$ be a ring. A non-empty subset $S$ of $R$ is said to be a multiplicative closed set in $R$ , if $a, b \in S$ implies $ab \in S$, and multiplicative closed set $S$ is called a multiplicative system if $0 \notin S$. Let $S$ be a multiplicative system in $R$ such that $[S, R] = \{0\}$, where $[S, R] = \{[s, r] | s \in S , r \in R\}$. Define a relation on $R \times S$ by $(a, s) \sim (b, t)$ if and only if there exists $x \in S$ such that $x(at - bs) = 0$. This is an equivalence relation on $R \times S$ and we denote the equivalence class of $(a, s)$ by $a_s$. Let $R_S$ denote the set of equivalence classes of $R \times S$ with respect to this relation, that is $R_S = \{a_s \mid (a, s) \in R \times S\}$ such that $a_s = \{(b, t) \in R \times S \mid (a, s) \sim (b, t)\}$. We can make $R_S$ in to ring by define addition (+) and multiplication(.): $a_s + b_t = (at + bs)_{st}$ and$a_s . b_t = (a. b)_{st}$, $for\ all\ a_s, b_t \in R_S$ which we call $(R_S, +, .)$ localization of $R$ at $S$ [1].

Throughout this paper $R$ is a ring with a nonzero center $Z(R)$. Recall that $R$ is prime if $aRb = \{0\}$ implies $a = 0\ or\ b = 0$, and it is said an n-torsion free, for $n \geq 1$, in case $na = 0, a \in R$ implies $a = 0$. As usual the commutator $ab - ba = 0$ will denoted by $[a, b]$ and anti-commutator $ab + ba$ will denoted by $a°b$. An additive mapping $d: R \longrightarrow R$ is called a derivation if
$d(ab) = d(a)b + ad(b)$, holds for all $a, b \in R$. An additive mapping $G: R \longrightarrow R$ is called a generalized derivation if there exists a derivation $d: R \longrightarrow R$ such that
$G(ab) = G(a)b + ad(b)$, holds for all $a, b \in R$.

Over the last several years, a number of authors studied the commutativity in prime ring admitting derivations and generalized derivation. In [2], Fillips proved that if $R$ is a prime ring with a nonzero ideal $I$ and $d$ is a derivation of $R$ such that $d([a, b]) = [a, b], for\ all\ a, b \in I$, then $R$ is commutative. In [3] Jabbar extended this result for centrally prime ring. Further Rehman [9] proved that $R$ is commutative if a generalized derivation $G$ associated with a nonzero derivation of a prime ring which acts as homomorphism or anti-homomorphism on a nonzero ideal $I$ of $R$. In [8], Quadri, Khan and Rehman they proved that if $R$ is a prime ring with a nonzero ideal $I$ and $G$ is a generalized derivation associated with a nonzero derivation $d$

of $R$ such that $G([a,b]) = [a,b]$ and $G(a\circ b) = a\circ b, for\ all\ a,b\ \in\ I$, then $R$ is commutative. In this present paper, we extend these results for generalized derivation on ideals in centrally prime ring.

## 1. Preliminaries.

We mention to some basic definitions and lemmas we needed in the prove of the main results:

**Definitions 1.1. [ 3]**

Let $R$ be a ring.Then

1- $R$ is called a centrally prime ring if $R_S$ is a prime ring for each multiplicative system $S$ in $R$ with $[S,R] = \{0\}$.

2- $R$ satisfies central commutation property (CCP) if $R_S$ is commutative for each multiplicative system $S$ in $R$ with $[S,R] = \{0\}$.

3- A mapping $f : R \rightarrow R$ is centrally-zero mapping on $R$ if $f(S) = \{0\}$ for each multiplicative system $S$ in $R$ with $[S,R] = \{0\}$.

We give an example for a centrally prime ring which is not prime.

**Example 1.2.** Let R={[0], [2], [4], [6]}. Then $(R, +_8, \cdot_8)$ is a ring. It is clear that R is a centrally prime ring but not prime.

**Remarks 1.3.** If $R$ is a ring and $S$ is a multiplicative system in $R$ such that $[S,R] = \{0\}$, then

i. If $s \in S$, then $s_s$ is the idenitity element of $R_S$ and $0_s$ is the zero of $R_S$, this identity does not depend on the choice of the elements of S, that is $s_{s=}t_t\ and\ 0_s = 0_t$, for all $s,t\ \in S$.

ii. If $a_s \in R_S$, where $a\ \in\ R$ and $s \in S$, then $(-a)_s$ is the additive inverse.

iii. If $a_s = 0$, where $a\ \in\ R$ and $s \in S$, then there exists $t \in S$ such that $ta = 0$.

iv. If $A \subseteq R$, then $A_S$ mean is the set $A_S = \{a_s \mid a \in A, s \in S\}$.

**Lemma 1.4.[3]** Let $R$ be a ring for which $Z(R)$ has no proper zero divisors of $R$. Then

1- $S = Z(R) - \{0\}$is multiplicative system in $R$ with $[S,R] = \{0\}$.

**Lemma 1.5.[4]** If $R$ is an n-torsion free ring and $S$ is a multiplicative system in $R$ such that $[S,R] = \{0\}$, then $R_S$ also is an n-torsion free ring.

**Lemma 1.6. [3]** Let $R$ be a ring for which $Z(R)$ has no proper zero divisors of $R$. Then

1- If $t \in Z(R) - \{0\}$ and $r \in\ R$ such that $tr = 0$ then $r = 0$.

2- If $R$ satisfies (CCP) then it is commutative.

**3-** $(Z(R))_S = Z(R_S)$, for all multiplicative systems $S$ in $R$ with $[S,R] = \{0\}$.

**Lemma 1.7.[7]** If a prime ring $R$ contains a nonzero commutative right ideal, then $R$ is commutative

**Theorem 1.8. [9]** Let $R$ be a 2-torsion free prime ring and $I$ be a nonzero ideal of $R$. Suppose $G: R \longrightarrow R$ is a nonzero generalized derivation associated with $d$.

(i) If $G$ acts as a homomorphism on $I$ and $d \neq 0$, then $R$ is commutative.

(ii) If $G$ acts as an anti- homomorphism on $I$ and $d \neq 0$, then $R$ is commutative

**Theorem 1.9.[8]** Let $R$ be a prime ring and $I$ be a nonzero ideal of $R$. If $R$ admits a generalized derivation $G$ associated with a nonzero derivation $d$ such that $G([a,b]) = [a,b]\ for\ all\ a,b\ \in\ I$, then $R$ is commutative.

**Theorem 1.10.[8]** Let $R$ be a prime ring and $I$ be a nonzero ideal of $R$. If $R$ admits a generalized derivation $G$ associated with a nonzero derivation $d$ such that $G(a\circ b) = a\circ b,\ for\ all\ a,b\ \in\ I$, then $R$ is commutative.

**Lemma 1.11. [3]** Let $R$ be a ring and $S$ a multiplicative system in $R$ such that $[S,R] = \{0\}$. If $d : R \rightarrow R$ be centrally zero derivation on $R$, then $d^*: R_S \longrightarrow R_S$ defined by $d^*(a_s) = (d(a))_s,\ for\ all\ a_s \in R_S$ is a derivation on $R_S$.

## 2. Main Results.

To prove our main theorem, we shall need the following lemmas**.**

**Lemma 2.1.** Let $R$ be a ring in which has no proper zero divisors and $S$ a multiplicative system in $R$ such that $[S,R]=\{0\}$. If $I$ is a nonzero ideal of $R$, then $I_S$ is a nonzero ideal of $R_S$.

**Proof.** Since $S \neq \emptyset$, then there exists $s \in S$ $and$ $0 \in I$, hence $0_S \in I_S$, this implies that $\emptyset \neq I_S \subseteq R_S$.

If $a_S,\ b_t \in I_S$, where $a, b \in I$ $and$ $s, t \in S$, then we have

$$a_S + \ b_t = (at + bs)_{st} \in I_S$$

and if $a_S \in I_S$ and $r_t \in R_S, for\ s, t \in S, r \in R\ and\ a \in I$, then we have

$$(a_S . r_t) = (ar)_{st} \in I_S\ and\ (r_t . a_s) = (ra)_{ts} \in I_S$$

Hence $I_S$ is an ideal of $R_S$.

If $I_S = 0$, hence $a_S = 0$, for $a \in I$ and $s \in S$, then there exists $t \in S$ such that $ta = 0$, but R has no proper zero divisor, so $a = 0\ or\ t = 0$, since $t \in S$ and $0 \notin S$, implies $t \neq 0$, thus by Lemma 1.6 we get $a = 0$, which is contradiction to $I \neq 0$. Therefore must be $I_S \neq 0$.

**Lemma 2.2. [5. Theorem 6 ]** Let $R$ be a ring which has no proper zero divisors. Then $R$ is centrally prime ring**.**

**Proof.** We will show that $R_S$ is prime.

**Lemma 2.3.** Let $R$ be a 2-torsion free centrally prime ring in which $Z(R)$ has no proper zero divisors and $I$ be a commutative centrally ideal of $R$. Then $R$ is commutative.

**Proof.** By Lemma 1.4, we have $S = Z(R) - \{0\}$ is a multiplicative system in $R$ with $[S,R] = \{0\}$.

By Lemma 1.5, we get $R_S$ is a 2-torsion free prime ring, and by Lemma 2.1, we get $I_S$ is a centrally ideal of $R_S$.

Since $I$ is commutative centrally ideal, then $I_S$ is also commutative since $(a_s b_t) = (a.b)_{st} = (ba)_{ts} = b_t . a_s, for\ all\ a, b \in R\ and\ s, t \in S.$

Hence by Lemma 1.7, we get $R_S$ is commutative therefore $R$ satisfies $CCP$. Hence $R$ is commutative.

**Lemma 2.4.** Let $R$ be a ring and $S$ a multiplicative system in $R$ such that $[S,R]=\{0\}$ and $F: R \to R$ be a generalized derivation with associated centrally zero derivation $d$. Then $G^*: R_S \longrightarrow R_S$ is a generalized derivation with associated derivation $d^*$on $R_S$.

**Proof.**

Define $G^*: R_S \longrightarrow R_S$ by $G^*(a_s) = (G(a))_s, for\ all\ a_s \in R_S,\ a \in R\ and\ s \in S$

First to show $G^*$ is well defined.

Now for all $a_{s,}\ b_t \in R_S$ such that $a_s = b_t$, the there exists $x \in S$ such that $(at - bs)x = 0$, so that $atx = bsx$.

Take generalized to both side, we get

$$G(at)x + atd(x) = G(bs)x + bsd(x)$$

Implies that $G(a)tx + ad(t)x = G(b)sx + bd(s)x$.

Hence, we get $G(a)tx = G(b)sx \Longrightarrow (G(a)t - G(b)s)x = 0 \Longrightarrow (G(a))_s = (G(b))_t$,

Therefore $G^*(a_s) = G^*(b_t)$.

Now we show that $G^*$ is generalized derivation associated with derivation $d^*$.

$$\begin{aligned} G^*(a_s + b_t) &= G^*((at + bs))_{st}) = (G(at + bs))_{st} = (G(at) + G(bs))_{st} \\ &= (G(a)t + ad(t) + G(b)s + bd(s))_{st} \\ &= (G(a)t + G(b)s)_{st} \\ &= (G(a)t)_{st} + (G(b)s)_{st} \\ &= (G(a))_s t_t + (G(b))_t\, s_s = G^*(a_s) + G^*(b_t) \end{aligned}$$

and

$$G^*(a_s . b_t) = G^*((a.b)_{st}) = \big(G(a.b)\big)_{st} = \big(G(a)b + ad(b)\big)_{st} = (G(a)b)_{st} + \big(ad(b)\big)_{st}$$

$$= (G(a))_s b_t + a_s (d(b))_t = G^*(a_s) b_t + a_s d^*(b_t).$$

Hence $G^*$ is generalized derivation associated with derivation $d^*$.

**Theorem 2.5.** Let $R$ be a 2-torsion free centrally prime ring in which $Z(R)$ has no proper zero divisors and $I$ be a nonzero centrally ideal of $R$. Suppose $G: R \longrightarrow R$ is a nonzero generalized derivation with associated centrally zero derivation $d$ and satisfying zero mapping.

(i) If $G$ acts as a homomorphism on $I$ and $d \neq 0$, then $R$ is commutative.

(ii) If $G$ acts as an anti- homomorphism on $I$ and $d \neq 0$, then $R$ is commutative

**Proof.** Since $R$ is a 2-torsion free centrally prime ring, then by Lemma 1.5, we have $R_S$ is a 2-torsion free prime ring, and since $I$ is a nonzero ideal of $R$, then by Lemma 2.1, we get $I_S$ is an ideal of $R_S$. By Lemma 1.4, we get $S = Z(R) - \{0\}$is a multiplicative system with $[S, R] = \{0\}$, that is $S \neq \emptyset$, fix an element $s \in S$. If $I_S = 0$, then for any $a \in I$, we have $a_s \in I_S$, such that $a_s = 0$, that is there exists $t \in S$ such that $ta = 0$, and $0 \neq t \in S \subseteq Z(R)$, but $Z(R)$ has no proper zero divisor of $R$, therefore , we get $a = 0$, which is contradiction to $I \neq 0$. Hence must be $I_S \neq 0$,.

If $d^* = 0$, fix an $s \in S$, then for any element $a \in I$, we have $a_s \in I_S$

So by Lemma 1.11 we have $(d(a))_s = d^*(a_s) = 0$, this implies that there exists $t \in S$ such that $td(a) = 0$. But $0 \neq t \in Z(R)$ and $Z(R)$ has no proper zero divisor of $R$. Therefore, we get $d(a) = 0$, which is contradiction for $d \neq 0$. Hence must be $d^* \neq 0$.

$(\boldsymbol{i})$ Since $G$ acts a homomorphism on $I$. Therefore, we must to show $G^*$ is acts a homomorphism on $I_S$.

Now let $a_s, b_t \in I_S, where\ a, b \in I\ and\ s, t \in S$. We have

$$G^*(a_s . b_t) = G^*((a.b)_{st}) = \big(G(a.b)\big)_{st} = \big(G(a).G(b)\big)_{st} = (G(a))_s.(G(b))_t$$
$$= G^*(a_s).G^*(b_t).$$

Hence $G^*$ acts as a homomorphism on $I_S$.

By Theorem 1.8 part (i), we have $R_S$ is a commutative, implies $R$ satisfies $CCP$. So we get $R$ is commutative ring.

$(\boldsymbol{ii})$Since $G$ acts anti- homomorphism on $I$, therefore, we must to show $G^*$ is acts anti- homomorphism on $I_S$.

Now let $a_s, b_t \in I_S, where\ a, b \in I\ and\ s, t \in S$. We have

$$G^*(a_s . b_t) = G^*((a.b)_{st}) = \big(G(a.b)\big)_{st} = \big(G(b).G(a)\big)_{st} = (G(b))_t.(G(a))_s$$
$$= G^*(b_t).G^*(a_s).$$

Hence $G^*$ acts as anti-homomorphism on $I_S$.

By Theorem 1.8 part (ii), we get $R_S$ is a commutative, implies $R$ satisfies $CCP$. Therefore $R$ is commutative ring.

**Theorem 2.6.** Let $R$ be a 2-torsion free centrally prime ring in which $Z(R)$ has no proper zero divisors and $I$ be a nonzero centrally ideal of $R$. If $G: R \longrightarrow R$ is a nonzero generalized derivation with associated centrally zero derivation $d$ such that $G([a, b]) = [a, b], for\ all\ a, b \in I,$ then $R$ is commutative.

**Proof**.

If $R$ is not satisfying (CCP), then there exists a multiplicative system $S$ in $R$ with $[S, R] = \{0\}$ such that $R_S$ is not commutative.

Since $R$ is centrally prime ring, then $R_S$ is prime ring.

Now we show that $I_S \neq 0 and\ d^* \neq 0$. If $I_S = 0$ and $S \neq \emptyset$, implies that there exists an element $s \in S$, so for any $b \in I$, we have $b_s = 0$, which implies there exists $t \in S$ such that $tb = 0$, where $0 \neq t \in S \subseteq Z(R)$. By Lemma 1.6, we get $b = 0$. Therefore $I = 0$, which is contradication. This implies $I_S \neq 0$.

Now if $d^* = 0$, we fix $s \in S$, since $S \neq \emptyset$, then for any element $a \in I$, we get $(d(a))_s = d^*(a_s) = 0, for\ all\ a_s \in I_S$ Thus there exists an element $x \in S$ such that $xd(a) = 0$, where $0 \neq x \in S \subseteq Z(R)$. Hence by Lemma 1.6, we get $d = 0$, which is contradiction, that is $d^* \neq 0$.

Now we must to show $G^*([a_s, b_t]) = ([a_s, b_t]), for\ all\ a_s, b_t \in I_S, where\ a, b \in I\ and\ s, t \in S$.

$$G^*([a_s . b_t]) = G^*(a_s . b_t - b_t . a_s)$$
$$= G^*((a.b)_{st} - (b.a)_{ts})$$
$$= G^*((a.b - b.a)_{st})$$
$$= G^*([a, b]_{st}) = (G([a, b])_{st}$$

$= ([a,b\,])_{st} = (a.b - b.a)_{st}$
$= ((a.b)_{st} - (b.a)_{st})$
$= (a_s.b_t - b_t.a_s) = [a_s.b_t]$

By Theorem 1.9, we get $R_S$ is a commutative, which is contradiction. Hence $R$ must be satisfies $(CCP)$ and $Z(R)$ has no proper zero divisor, we get $R$ is commutative ring.

**Theorem 2.7.** Let $R$ be a 2-torsion free centrally prime ring in which $Z(R)$ has no proper zero divisors and $I$ be a nonzero centrally ideal of $R$. If $G: R \longrightarrow R$ is a nonzero generalized derivation with associated centrally zero derivation $d$ such that $G(a^\circ b) = a^\circ b, for\ all\ a, b \in I,$ then $R$ is commutative.

**Proof**.

If $R$ is not satisfying $(CCP)$, then there exists a multiplicative system $S$ in $R$ with $[S,R] = \{0\}$ such that $R_S$ is not commutative.

Since $R$ is centrally prime ring so is $R_S$, then $R_S$ is prime.

Now we show that $I_S \neq 0 and\ d^* \neq 0$. If $I_S = 0$, then $S \neq \emptyset$, implies that there exists an element $s \in S$, so for any $b \in I$, we have $b_s = 0$, which implies there exists $t \in S$ such that $tb = 0$, where $0 \neq t \in S \subseteq Z(R)$. By Lemma 1.6, we get $b = 0$. Therefore $I = 0$, which is contradication. This implies $I_S \neq 0$.

Now if $d^* = 0$, we fix $s \in S$ ( since $S \neq \emptyset$), then for any element $a \in I$, we get $(d(a))_s = d^*(a_s) = 0, for\ all\ a_s \in I_S$. Thus there exists an element $x \in S$ such that $xd(a) = 0$, where $0 \neq x \in S \subseteq Z(R)$. Hence by Lemma 1.6, we get $d = 0$, which is contradiction, that is $d^* \neq 0$.

Now we must to show $G^*(a_s{}^\circ\, b_t) = (a_s{}^\circ\, b_t), for\ all\ a_s, b_t \in I_{S,} where\ a, b \in I\ and\ s, t \in S.$

$G^*(a_s{}^\circ\, b_t) = G^*(a_s.b_t + b_t.a_s)$
$= G^*((a.b)_{st} + (b.a)_{ts})$
$= G^*((a.b + b.a)_{st})$
$= G^*((a^\circ b)_{st}) = (G(a^\circ b)_{st}$
$= (a^\circ b)_{st} = (a.b + b.a)_{st}$
$= ((a.b)_{st} + (b.a)_{st})$
$= (a_s.b_t + b_t.a_s) = a_s{}^\circ\, b_t$

By Theorem 1.10, we get $R_S$ is a commutative. Which is contradiction, therefore must be $R$ satisfies CCP and $Z(R)$ has no proper zero divisor, we get $R$ is commutative..

## References


[1] Fahr D.P. *"Skew Fields of Fractions"*. M.Sc project. Department of Mathematics, University College London. (2002).

[2] Filippis V.D. *"Automorphisms and derivations in prime rings"*. Rendiconti di Matermatica, Roma, serie VII, Vol. 19, pp 393-404. (1999).

[3] Jabbar A. K. *"Centrally prime rings which are commutative"*. J.Kirkuk Univ. Scienctific Studies. Vol. 1, No. 2, pp 108-124. (2006).

[4] Jabbar A. K. *"Some properties of centrally closed Lie ideals of centrally prime rings*". J. Al-Nahrain Univ. Vol. 10, No. 2, pp 162-167. (2007).

[5] Jabbar A. K. & Majeed A. H. *"On Centrally prime and Centrally semiprime rings"*. Raf. J. of Comp. & Math's. Vol. 5, No. 1, pp 47-56. (2008).

[6]Larsen M. D. & McCarthy P.J. *"Multiplicative theory of Ideals"*. Academic Press New York and London, pp 62. (1971).

[7] Mayn J.H. "*Centralizing mappings of prime rings*". Candan. Math. Bull. Vol. 27, pp 122-126. (1984).

[8] Ouadri M. Khan A., M. S. & Rehman N. *"Generalized derivations and commutativity of prime rings"*. Indian J. pure appl. Math. Vol. 34, No. 9, pp 1393-1396. (2003).

[9] Rehman N. "*On generalized derivations as homomorphisms and anti-homomorphism*". Glasnik Matematicki, Vol. 39, No. 1, pp 27-30. (2004).